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Dirac operator, bicovariant differential calculus and gauge theory on κ-Minkowski space

P N Bibikov

St Petersburg Branch of Steklov Mathematical Institute, Fontanka 27, St Petersburg, 191011, Russia

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Abstract. Connections between the κ-Poincaré covariant space Γ of differential 1-forms on κ-Minkowski space, Dirac operator and Alain Connes formula are studied. The equations and Lagrangian of gauge theory are constructed. The appearance of an additional spin-0 gauge field according to the non-trivial structure of Γ is studied.

0. Introduction

Recently, non-commutative geometry [1] has attracted a great deal of interest from many researchers as a natural framework for quantization of space and time. One of the most promising results in this direction is the approach to gauge field theory developed in [2] where the standard model of gauge interaction was obtained from non-commutativity of spacetime. A review of different deformations of Minkowski space which are connected with the corresponding deformations of Lorentz and Poincaré groups is given in the review [12].

The basic notion of the approach studied in [1, 2] is the Connes triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is in the general framework a non-commutative *-algebra which is considered as an algebra of operators in the Hilbert space $\mathcal{H}$. D is a linear, possibly unbounded, operator in $\mathcal{H}$ with $D^* = -D$.

In the classical case when $\mathcal{A} = \mathrm{Fun}(M)$ is a commutative algebra of functions on the differential manifold M and the operator D is the usual Dirac operator

$$D = \gamma^i \partial_i. \tag{0.1}$$

In (0.1) ∂_i are local derivatives and for each $x \in M$ matrices $\gamma^i(x)$ satisfying relations

$$\gamma^i(x)\gamma^j(x) + \gamma^j(x)\gamma^i(x) = 2g^{ij}(x) \tag{0.2}$$

are generators of a local Clifford algebra $Cl(x)$. In equation (0.2) $g^{ij}(x)$ are local components of the metric tensor.

The vector bundle over M whose fibre over each point $x \in M$ is an algebra $Cl(x)$ is called the Clifford bundle over M [4]. This bundle was associated in [1, 2] with the space of all quantum differential forms over M; however, the space of all 1-forms is a subbundle in $Cl(M)$, whose fibre over each point $x \in M$ is generated as a linear space by elements $\gamma^i(x)$. We shall call it the Dirac bundle and denote it by $\mathrm{Dir}(M, D)$.

Non-commutative differential calculus on the Fun(M) is defined by introduction of the exterior derivative operator which we shall denote by $\mathrm{d_c}$. For each $f \in \mathrm{Fun}(M)$ it has the form

$$\mathrm{d_c} f = [D, f]. \tag{0.3}$$

According to (0.1), formula (0.3) gives the following result:

$$\mathrm{d_c} f = \partial_i(f)\gamma^i. \tag{0.4}$$

Correspondence of the definition (0.3) with the usual external derivative

$$\mathrm{d} f = \partial_i(f)\,\mathrm{d}x^i \tag{0.5}$$

follows from the isomorphism between $T^*(M)$ and $\mathrm{Dir}(M, D)$ or in an algebraic framework between their spaces of sections Γ and Dir(Fun(M), D), where Γ is the space of differential 1-forms over M or the space of sections of the cotangent bundle $T^*(M)$ and Dir(Fun(M), D) is the space of sections of $\mathrm{Dir}(M, D)$. The isomorphism follows from the fact that all fibres of $T^*(M)$ and $\mathrm{Dir}(M, D)$, as well as the corresponding gluing maps, are isomorphic. This isomorphism may be expressed by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Fun}(M) & \stackrel{\mathrm{d}f=\partial_i(f)\,\mathrm{d}x^i}{\longrightarrow} & \Gamma \\ \downarrow \mathrm{id} & & \downarrow \mathrm{d}x^i \to \gamma^i \\ \mathrm{Fun}(M) & \stackrel{\mathrm{d_c}f=[D,f]}{\longrightarrow} & \mathrm{Dir}(\mathrm{Fun}(M), D) \end{array} \tag{0.6}$$

As a bimodule over Fun(M), Dir(Fun(M), D) is generated by all sums of the form

$$\sum_i f_i[D, g_i]. \tag{0.7}$$

According to the isomorphism (0.6) the gauge connection 1-form $A_i\,\mathrm{d}x^i$, which is used in construction of the pure gauge action, and the gauge interaction term $A_i\gamma^i$ in the Dirac equation for the spinor field have similar geometrical interpretations. Therefore, when studying the deformations of field theory in quantum spaces, it is natural to suppose that the diagram (0.6) also has an analogue in the non-commutative case. We shall write the corresponding diagram in the form,

$$\begin{array}{ccc} \mathcal{A} & \stackrel{\mathrm{d}f=\mathrm{d}f}{\longrightarrow} & \Gamma \\ \downarrow \mathrm{id} & & \downarrow \mathrm{d}f \to \mathrm{d_c} f \\ \mathcal{A} & \stackrel{\mathrm{d_c}f=[D,f]}{\longrightarrow} & \mathrm{Dir}(\mathcal{A}, D) \end{array} \tag{0.8}$$

where $\mathrm{Dir}(\mathcal{A}, D)$ is a bimodule over $\mathcal{A}$ generated by all sums of the form (0.7).

By its meaning the diagram (0.8) is much richer than the rather tautological diagram (0.6) because in quantum cases elements of Γ as well as elements of $\mathrm{Dir}(\mathcal{A}, D)$ have non-trivial commutation relations with elements of $\mathcal{A}$, so that the dependence of $\mathrm{Dir}(\mathcal{A}, D)$ on D becomes more essential.

When the Dirac operator is defined, formula (0.3) gives an explicit construction for the external differential $\mathrm{d_c}$. However, for the most interesting class of non-commutative spaces which appears in applications of quantum group theory [14], the non-commutative differential calculus may be defined in the purely abstract form [3]. So in this case the commutativity of the diagram (0.8) means an equivalence of the two approaches for construction of the non-commutative differential calculus according to [1] or [3]. In the paper we show that this commutativity may be used as a powerful tool for constructing the corresponding Dirac operator.

In the framework of [1, 2], the Dirac equation for a massless spinor field coupled with the gauge potential has the form

$$(D + V)\psi = 0 \tag{0.9}$$

where $\psi \in \mathcal{H}$ and V is a non-commutative analogue of $\mathrm{i}gA_i\gamma^i$ where g is a gauge charge. According to the isomorphism between the quantum Dirac and the quantum cotangent bundles supposed by (0.8), it corresponds to the gauge connection quantum 1-form ω, which is the non-commutative analogue of $\mathrm{i}gA_i\,\mathrm{d}x^i$.

We take the gauge transformation law for spinorial fields in the form

$$\psi \to U\psi \tag{0.10}$$

where U is a unitary element of $\mathcal{A}$

$$UU^* = U^*U = 1. \tag{0.11}$$

(An additional restriction on U will be discussed in section 3). The transformation (0.10) for ψ is compatible with the following transformation for V

$$V \to UVU^* + U[D, U^*] \tag{0.12}$$

which, according to (0.8), is equivalent to the following law for ω:

$$\omega \to \tilde{\omega} = U\omega U^* + U\,\mathrm{d}U^*. \tag{0.13}$$

In the present paper we consider the Dirac operator for the κ-Minkowski space $\mathcal{M}_\kappa$ as one of the most studied deformations of the usual Minkowski space [5–8]. On $\mathcal{M}_\kappa$ may be defined the right or left coaction of the quantum Poincaré group $\mathcal{P}_\kappa$. The left $\mathcal{P}_\kappa$-covariant differential calculus on $\mathcal{M}_\kappa$ was defined in [6]. The Dirac operator on $\mathcal{M}_\kappa$ was proposed in [11]. In this paper we (using left analogues of [6–8] constructions) define on $\mathcal{M}_\kappa$ Dirac operator which has a more general form than that defined in [11], but in some special case it coincides with the latter. In contrast with the approach used in [11], we construct the Dirac operator according to the condition of commutativity of the diagram (0.8). Our construction has a general form and may be also used for different quantum spaces admitting the quantum group coaction. A right (or left) covariant differential calculus on these spaces can be constructed according to [3]. As will be mentioned in the conclusions, the commutativity of diagram (0.8) follows from the general formulae of quantum differential calculus.

A description of different Minkowski space deformations has been given in the review in [12]. (Examples of Dirac operators on the quantum $SU(2)$ group and the quantum sphere have been discussed in [9] and [13].)

The paper is organized as follows. In section 1 we study according to [6–8] the differential geometry on $\mathcal{M}_\kappa$. According to [15], the algebra of differential operators on $\mathcal{M}_\kappa$ is defined as the unified algebra consisting of both the elements of $\mathcal{M}_\kappa$ and its Hopf dual $\mathcal{M}_\kappa^*$, with commutation relations between them induced by the left-invariant action of $\mathcal{M}_\kappa^*$ on $\mathcal{M}_\kappa$.

It is shown that on this algebra a right $\mathcal{P}_\kappa$ coaction may be defined. The corresponding generators on which it has the simplest form are presented. The invariant Klein–Gordon operator on $\mathcal{M}_\kappa$ is constructed as a bilinear combination of these generators.

Construction of the exterior differential needs the introduction of quantum derivatives which are also elements of the quantum Poincaré algebra. In section 2, according to diagram (0.8), we construct an $\mathcal{M}_\kappa$ Dirac operator. In section 3 we derive in various forms the equations of deformed electrodynamics on $\mathcal{M}_\kappa$. We also define the deformed Lagrangian. However, in the non-commutative case there is no such correspondence between the Lagrangian and the equations of motion. Moreover, the gauge invariance group of these equations is much larger than the corresponding group for the Lagrangian. We also mention that the existence of an additional dimension of the space of the quantum differential 1-forms leads to the natural appearance of the extra spin-0 gauge field.

Everywhere in this paper we use the Einstein rules of summation. In the first section, according to the notation used in [8], we suppose that the greek indices α, μ, ν numerate the spacetime components and take the values $0, 1, 2, 3$; however, the latin indices m and n numerate only the space components and take values $1, 2, 3$. Throughout, $g^{\mu\nu}$ means the Minkowski space metric tensor $(1, -1, -1, -1)$.

1. κ-Poincaré group and κ-Minkowski space

The κ-Poincaré quantum group $\mathcal{P}_\kappa$ was introduced in [7] (see also [6, 8]) and in one of the equivalent forms it represents as a *-Hopf algebra generated by Hermitian elements ${\Lambda_\mu}^\nu$, a^μ and relations

$$\begin{aligned}
&[a^\mu, a^\nu] = \frac{\mathrm{i}}{\kappa}(\delta_0^\mu a^\nu - \delta_0^\nu a^\mu)\\
&[{\Lambda_\mu}^\nu, {\Lambda_\alpha}^\beta] = 0\\
&[{\Lambda_\mu}^\nu, a^\alpha] = \frac{\mathrm{i}}{\kappa}[(\delta_0^\nu - {\Lambda_0}^\nu){\Lambda_\mu}^\alpha + g^{\nu\alpha}(\delta_\mu^0 - {\Lambda_\mu}^0)]\\
&\triangle({\Lambda_\mu}^\nu) = {\Lambda_\mu}^\alpha \otimes {\Lambda_\alpha}^\nu\\
&\triangle(a^\mu) = a^\nu \otimes {\Lambda_\nu}^\mu + 1 \otimes a^\mu\\
&S({\Lambda_\mu}^\nu) = {\Lambda^\nu}_\mu = g_{\mu\alpha} g^{\nu\beta} {\Lambda_\beta}^\alpha\\
&S(a^\mu) = -a^\nu {\Lambda^\mu}_\nu\\
&\varepsilon({\Lambda_\mu}^\nu) = \delta_\mu^\nu\\
&\varepsilon(a^\mu) = 0.
\end{aligned} \tag{1.1}$$

The $\mathcal{P}_\kappa$ may be regarded as the quantum symmetry group of the κ-Minkowski space $\mathcal{M}_\kappa$, which is defined by four Hermitian generators x^μ and the relations

$$[x^\mu, x^\nu] = \frac{\mathrm{i}}{\kappa}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu). \tag{1.2}$$

The corresponding right $\mathcal{P}_\kappa$ coaction is

$$\Phi_{\mathrm{R}}(x^\mu) = x^\nu \otimes {\Lambda_\nu}^\mu + 1 \otimes a^\mu \tag{1.3}$$

(the left comodule structure can also be defined [6, 7]).

The coproduct, counit, and antipode [8]

$$\begin{aligned}
&\triangle(x^\mu) = 1 \otimes x^\mu + x^\mu \otimes 1\\
&\varepsilon(x^\mu) = 0\\
&S(x^\mu) = -x^\mu
\end{aligned} \tag{1.4}$$

also define on $\mathcal{M}_\kappa$ the structure of the cocommutative Hopf algebra. As was shown in [8], the correspondence $a^\mu \to x^\mu$ and ${\Lambda^\mu}_\nu \to \delta_\nu^\mu$ defines a Hopf algebra homomorphism from $\mathcal{P}_\kappa$ to $\mathcal{M}_\kappa$.

Its Hopf dual $\mathcal{M}_\kappa^*$ is defined by four Hermitian generators P_μ and the relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0\\
&\triangle(P_0) = P_0 \otimes 1 + 1 \otimes P_0\\
&\triangle(P_m) = P_m \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes P_m
\end{aligned}$$

$$S(P_0) = -P_0$$
$$S(P_m) = -\mathrm{e}^{P_0/\kappa} P_m$$
$$\varepsilon(P_\mu) = 0. \tag{1.5}$$

The pairing $(\cdot, \cdot) : \mathcal{M}_\kappa^* \otimes \mathcal{M}_\kappa \to \mathbb{C}$ is given by

$$\mathrm{i}(P_\mu, x^\nu) = \delta_\mu^\nu. \tag{1.6}$$

As was shown in [8], the algebra $\mathcal{M}_\kappa^*$ is a Hopf subalgebra of the quantum Poincaré algebra which is dual to $\mathcal{P}_\kappa$. The corresponding pairing between $\mathcal{M}_\kappa^*$ and $\mathcal{P}_\kappa$ is given by the formula

$$\mathrm{i}(P_\mu, a^\nu) = \delta_\mu^\nu. \tag{1.7}$$

According to this pairing, $\mathcal{M}_\kappa^*$ acts on $\mathcal{M}_\kappa$ from the left

$$\pi(x) = ((\mathrm{id} \otimes \pi), \Phi_{\mathrm{R}}(x)) \qquad \pi \in \mathcal{M}_\kappa^*, \quad x \in \mathcal{M}_\kappa. \tag{1.8}$$

Considering the elements of $\mathcal{M}_\kappa$ as the left multiplication operators we may obtain, according to (1.5), the following relations:

$$[P_0, x^\mu] = \frac{1}{\mathrm{i}} \delta_0^\mu$$
$$[P_m, x^0] = \frac{\mathrm{i}}{\kappa} P_m$$
$$[P_m, x^n] = \frac{1}{\mathrm{i}} \delta_m^n. \tag{1.9}$$

The elements

$$e^4 = \mathrm{i}\kappa \left(\mathrm{ch} \frac{P_0}{\kappa} - \frac{1}{2\kappa^2} \mathrm{e}^{P_0/\kappa} \boldsymbol{P}^2 \right)$$
$$e^0 = \mathrm{i}\kappa \left(\mathrm{sh} \frac{P_0}{\kappa} + \frac{1}{2\kappa^2} \mathrm{e}^{P_0/\kappa} \boldsymbol{P}^2 \right)$$
$$e^m = -\mathrm{i}\mathrm{e}^{P_0/\kappa} P_m \tag{1.10}$$

satisfy the following commutation relations with the elements of $\mathcal{M}_\kappa$,

$$[e^\mu, x^\nu] = \frac{\mathrm{i}}{\kappa} (g^{0\mu} e^\nu - g^{\mu\nu} e^0 - g^{\mu\nu} e^4)$$
$$[e^4, x^\mu] = -\frac{\mathrm{i}}{\kappa} e^\mu \tag{1.11}$$

and the additional relation

$$\Box_\kappa \equiv e_\mu e^\mu = g^{\mu\nu} e_\mu e_\nu = \kappa^2 + (e^4)^2. \tag{1.12}$$

Equations (1.11) and (1.12) are invariant under the right $\mathcal{P}_\kappa$ coaction which on $\mathcal{M}_\kappa$ has the form (1.3) and on the elements (1.10) is defined by

$$\Phi_{\mathrm{R}}(e_\mu) = e_\nu \otimes \Lambda^\nu{}_\mu \qquad \Phi_{\mathrm{R}}(e^4) = e^4 \otimes 1. \tag{1.13}$$

The proof of this statement can be easily obtained by comparing equations (1.11) and (1.13) with analogous formulae (1.15) and (1.17) taken from [6] and given below. These formulae correspond to the space of differential 1-forms but have the same form as (1.11) and (1.13). The invariance of (1.12) under (1.13) may be proved easily by direct calculation.

We may also interpret formulae (1.13) together with (1.3) as a natural $\mathcal{P}_\kappa$ coaction on the algebra of differential operators.

So, according to the general approach [10, 15] we may consider the joint algebra generated by x^μ, e_μ, e^4 and relations (1.2), (1.11) and (1.12) as the algebra of differential operators on $\mathcal{M}_\kappa$.

Element $\Box_\kappa$ from (1.12) is invariant under (1.13) and we may consider it as a massless Klein–Gordon operator on $\mathcal{M}_\kappa$. (It can be expressed from the Klein–Gordon operator C_1 suggested in [11] by the relation $\Box_\kappa = C_1(1 + (1/4\kappa^2)C_1)$.)

Following [6] we define the quantum De Rham complex on $\mathcal{M}_\kappa$ (the Leibniz rule is satisfied). As a $\mathcal{M}_\kappa$ bimodule the space of 1-forms Γ is generated by $\tau^\mu = \mathrm{d}x^\mu$. The commutation relations between the elements of Γ and $\mathcal{M}_\kappa$ may be written using an additional 1-form [6]:

$$\tau^4 = \frac{\mathrm{i}\kappa}{4}[\tau^\mu, x_\mu] - \frac{3}{4}\tau^0. \tag{1.14}$$

These relations have the form [6]

$$[\tau^\mu, x^\nu] = \frac{\mathrm{i}}{\kappa}(g^{0\mu}\tau^\nu - g^{\mu\nu}\tau^0 - g^{\mu\nu}\tau^4) \qquad [\tau^4, x^\mu] = -\frac{\mathrm{i}}{\kappa}\tau^\mu. \tag{1.15}$$

The external algebra relations and the external derivative are given by ($i, j = 0, \ldots, 4$)

$$\tau^i \wedge \tau^j = -\tau^j \wedge \tau^i \qquad \mathrm{d}\tau^i = 0. \tag{1.16}$$

Equations (1.14)–(1.16) are invariant under the right $\mathcal{P}_\kappa$-coaction given on the elements of $\mathcal{M}_\kappa$ by (1.3) and on the elements Γ by

$$\Phi_{\mathrm{R}}(\tau^\mu) = \tau^\nu \otimes \Lambda_\nu{}^\mu \qquad \Phi_{\mathrm{R}}(\tau^4) = \tau^4 \otimes 1. \tag{1.17}$$

(In [6] the left variant of (1.17) was presented.)

It is easy to see from (1.15) that for every $a \in \mathcal{M}_\kappa$

$$s^2 a = a s^2 \qquad \tau^5 a = a\tau^5 \tag{1.18}$$

where the metric form $s^2 \in \Gamma \otimes \Gamma$ and the volume form $\tau^5 \in \Gamma^{\wedge 5}$ are defined by

$$s^2 = \tau_\mu \otimes \tau^\mu - \tau^4 \otimes \tau^4 \qquad \tau^5 = \tau^0 \wedge \tau^1 \wedge \tau^2 \wedge \tau^3 \wedge \tau^4. \tag{1.19}$$

These forms are invariant under the right $\mathcal{P}_\kappa$ coaction on $\Gamma \otimes \Gamma$ and $\Gamma^{\wedge 5}$.

According to (1.12) and (1.18), we suggest the following components of the metric tensor

$$g^{44} = g_{44} = -1 \tag{1.20}$$

corresponding to the e^4 and τ^4.

The commutation relations between the 1-forms and elements of $\mathcal{M}_\kappa$ may also be represented in the standard form [3] ($i, j = 0, 1, 2, 3, 4$)

$$\tau^i a = f^i{}_j(a)\tau^j \tag{1.21}$$

where $f^i{}_k$ are linear operators $f^i{}_k : \mathcal{M}_\kappa \to \mathcal{M}_\kappa$. From $\tau^i(ab) = (\tau^i a)b$ it follows that

$$f^i{}_k(ab) = f^i{}_j(a) f^j{}_k(b). \tag{1.22}$$

In the most interesting case when all $f^i{}_j \in \mathcal{M}_\kappa^*$, so that their action on elements of $\mathcal{M}_\kappa$ is given by (1.8), this is equivalent to

$$\triangle(f^i{}_k) = f^i{}_j \otimes f^j{}_k. \tag{1.23}$$

Taking

$$
\begin{aligned}
&f^0{}_0 = \mathrm{ch}\frac{P_0}{\kappa} + \frac{1}{2\kappa^2}\mathrm{e}^{P_0/\kappa}\boldsymbol{P}^2 \\
&f^0{}_m = -\frac{1}{\kappa}P_m \qquad f^m{}_0 = -\frac{1}{\kappa}\mathrm{e}^{P_0/\kappa}P_m \qquad f^n{}_m = \delta^n_m \\
&f^0{}_4 = \mathrm{sh}\frac{P_0}{\kappa} + \frac{1}{2\kappa^2}\mathrm{e}^{P_0/\kappa}\boldsymbol{P}^2 \\
&f^4{}_0 = \mathrm{sh}\frac{P_0}{\kappa} - \frac{1}{2\kappa^2}\mathrm{e}^{P_0/\kappa}\boldsymbol{P}^2 \\
&f^m{}_4 = -\frac{1}{\kappa}\mathrm{e}^{P_0/\kappa}P_m \qquad f^4{}_m = \frac{1}{\kappa}P_m \\
&f^4{}_4 = \mathrm{ch}\frac{P_0}{\kappa} - \frac{1}{2\kappa^2}\mathrm{e}^{P_0/\kappa}\boldsymbol{P}^2
\end{aligned} \tag{1.24}
$$

we can directly prove the commutation relations (1.15) and the coaction formulae (1.23).

From (1.15) and (1.17) it follows that the correspondence $\star : \Gamma^{\wedge 2} \to \Gamma^{\wedge 3}$ defined by the formulae

$$
\begin{aligned}
&\star\tau^4 \wedge \tau^0 = \tau^1 \wedge \tau^2 \wedge \tau^3 \\
&\star\tau^4 \wedge \tau^1 = \tau^0 \wedge \tau^2 \wedge \tau^3 \\
&\star\tau^0 \wedge \tau^1 = \tau^4 \wedge \tau^2 \wedge \tau^3 \\
&\star\tau^1 \wedge \tau^2 = -\tau^4 \wedge \tau^0 \wedge \tau^3.
\end{aligned} \tag{1.25}
$$

and their cyclic permutations is a *-homomorphism. It agrees with the right $\mathcal{P}_\kappa$-comodule and $\mathcal{M}_\kappa$ bimodule structures on $\Gamma^{\wedge 2}$ and $\Gamma^{\wedge 3}$. We shall need this homomorphism in section 3 to construct the equations and Lagrangian of gauge theory.

From the definition of τ^5 (1.19) and the commutation relations (1.15) it follows that for every $\alpha \in \Gamma^{\wedge 2}$

$$\alpha \wedge \star\alpha = \star\alpha \wedge \alpha. \tag{1.26}$$

We shall use this property in the definition of the gauge invariant Lagrangian (3.17).

2. Dirac operator and differential calculus

Let us now define the following elements of $\mathcal{M}^*_\kappa$:

$$\partial_0 = \mathrm{i}\kappa f^4{}_0 \qquad \partial_m = \mathrm{i}\kappa f^4{}_m \qquad \partial_4 = \mathrm{i}\kappa(f^4{}_4 - 1). \tag{2.1}$$

Using these elements we may write the formula for the external derivation in the compact form ($i = 0, 1, 2, 3, 4$)

$$\mathrm{d}a = \partial_i(a)\tau^i. \tag{2.2}$$

According to the Leibniz rule

$$\mathrm{d}(ab) = a\,\mathrm{d}b + \mathrm{d}a\,b \tag{2.3}$$

the following system of relations must be satisfied ($i, j = 0, 1, 2, 3, 4$):

$$\partial_i(ab) = a\partial_i(b) + \partial_j(a)f^j{}_i(b) \tag{2.4}$$

or

$$[\partial_i, a] = \partial_j(a)f^j{}_i. \tag{2.5}$$

Since all $\partial_i \in \mathcal{M}_\kappa^*$ this is equivalent to

$$\triangle(\partial_i) = 1 \otimes \partial_i + \partial_j \otimes f^j{}_i. \tag{2.6}$$

So to prove (2.2) we must check it for the coordinates x^μ and then prove (2.6), which can be done easily by strict calculations.

We take the Dirac operator in the form ($i = 0, \ldots, 4$)

$$D_\kappa = \gamma^i \partial_i \tag{2.7}$$

where γ^i, for $i = 0, \ldots, 3$, are the usual Dirac gamma matrices satisfying the standard relation

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2g^{ij} \tag{2.8}$$

(which is a standard Minkowski space metric) and γ_4 is some undefinite matrix which, however, may be taken in the form $\gamma_4 = \lambda I_4$ where I_4 is a unit 4×4 matrix or $\gamma_4 = \lambda\gamma_5$ where $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$. The choice $\gamma_4 = 0$ corresponds to the Dirac operator suggested in [11].

In this case, the connection between D_κ and $\Box_\kappa$ has the standard form

$$D_\kappa^2 = \Box_\kappa. \tag{2.9}$$

How to define the Hilbert space $\mathcal{H}$ correctly, where the operator D_κ acts, is a problem. According to the general approach [1, 2] it should be of the form $\mathbb{C}^4 \otimes \mathcal{M}_\kappa^{\mathrm{reg}}$, where the Hilbert space $\mathcal{M}_\kappa^{\mathrm{reg}}$ is a Hilbert completion of an appropriate subalgebra in $\mathcal{M}_\kappa$. This statement follows from the commutative limit $\kappa \to \infty$ where $\mathcal{H} = \mathbb{C}^4 \otimes L^2(M^4)$.

The direct application of (0.3) gives, according to (2.5), the following expressions for τ_{c}^i corresponding to the τ^i elements of $D\mathrm{ir}(\mathcal{M}_\kappa, D_\kappa)$

$$\tau_{\mathrm{c}}^i = \gamma^j f^i{}_j. \tag{2.10}$$

According to (1.22), relations (1.21) are also fulfilled for τ_{c}^i and the diagram (0.8) is commutative. As we see this fact does not depend on the explicit form of γ^4 (as well as on the other γ-matrices). Generally, it expresses the statement that on the definition (2.7) derivatives ∂_i must be chosen in accordance with the commutativity diagram (0.8); however, algebraical properties of γ-matrices lead to connection between the Dirac and Klein–Gordon operators. We shall discuss an additional role of γ^4 in the next section.

3. Gauge theory on $\mathcal{M}_\kappa$

In this section we suppose that all up and down indices take values from 0 to 4.

By analogy with the classical case we define the gauge potentials as elements of $\mathcal{M}_\kappa$, the quantum algebra of functions. Let us introduce the $U(1)$ gauge field by the anti-Hermitean gauge connection 1-form

$$\omega = \mathrm{i}A_k \tau^k \tag{3.1}$$

where A_k, $k = 0, \ldots, 3$, are deformations of the usual potential and A_4 may be interpreted as a spin-0 gauge field. The appearance of such scalar gauge fields in the framework of non-commutative geometry has been intensively studied in [1] and [2].

The condition that ω must be anti-Hermitean is equivalent to

$$f^i{}_j(A_i^*) = A_j. \tag{3.2}$$

It seems important that equation (3.2) mixes A_μ and A_4.

According to (0.9) and (2.7) and connection between ω and V, we can write the gauge coupled Dirac equation for a massless particle in the form

$$\gamma^k \bigtriangledown_k \psi = 0 \tag{3.3}$$

where

$$\bigtriangledown_k = \partial_k + \mathrm{i} g A_j f^j{}_k \tag{3.4}$$

(g is a gauge charge).

The transformation law (0.13) gives

$$\tilde{A}_k = U A_j f^j{}_k(U^*) - \mathrm{i}/g U \partial_k(U^*). \tag{3.5}$$

Defining the anti-Hermitean curvature form

$$\Omega = \mathrm{d}\omega + g\omega \wedge \omega \tag{3.6}$$

we obtain, according to (0.13), the following transformation law:

$$\tilde{\Omega} = U\Omega U^*. \tag{3.7}$$

Defining the field strength tensor by

$$\mathrm{i} F_{jk} \tau^j \wedge \tau^k = \Omega \tag{3.8}$$

or according to (1.16)

$$F_{ij} = \partial_i(A_j) - \partial_j(A_i) + \mathrm{i} A_k [f^k{}_i(A_j) - f^k{}_j(A_i)] \tag{3.9}$$

we can use relations (3.7), (1.21) and (1.22) to obtain the following transformation law:

$$\tilde{F}_{ij} = U F_{kl} f^k{}_i f^l{}_j(U^*). \tag{3.10}$$

We may also obtain the tensor F_{ij} by commuting covariant derivatives

$$[\bigtriangledown_i, \bigtriangledown_j] = \mathrm{i} g F_{mn} f^m_i f^n{}_j. \tag{3.11}$$

It is easy to prove that the following Bianchi identities

$$[\bigtriangledown_i, [\bigtriangledown_j, \bigtriangledown_k]] + [\bigtriangledown_k, [\bigtriangledown_i, \bigtriangledown_j]] + [\bigtriangledown_j, [\bigtriangledown_k, \bigtriangledown_i]] = 0 \tag{3.12}$$

are satisfied.

Defining the deformed covariant derivatives of the strength tensor as

$$\bigtriangledown_m F^{mk} = \partial_m F^{mk} + \mathrm{i} g (A_j f^j{}_m(F^{mk}) - F^{mn} f_m{}^j f_n{}^k(A_j)) \tag{3.13}$$

it is easy to obtain the following transformation law:

$$\tilde{\bigtriangledown}_m F^{mk} = U \bigtriangledown_m F^{mn} f_n{}^k(U^*). \tag{3.14}$$

In the limit $\kappa \to \infty$, it follows from (1.24) (or generally from (1.21)) that we have $f^m_n \to \delta^m_n$ so that the transformation laws (3.10) and (3.14) may be considered as deformations of the standard formulae. Equations (3.12) and

$$\bigtriangledown_m F^{mk} = 0 \tag{3.15}$$

may be interpreted as $\mathcal{P}_\kappa$-covariant equations of deformed electrodynamics in κ-Minkowski space.

We may also represent equations (3.12) and (3.15) in a purely geometrical form using the deformed covariant derivatives and the homomorphism $\star$:

$$\begin{aligned} D\Omega &= \mathrm{d}\Omega + \omega \wedge \Omega - \Omega \wedge \omega = 0 \\ D \star \Omega &= \mathrm{d} \star \Omega + \omega \wedge \star\Omega - \star\Omega \wedge \omega. \end{aligned} \tag{3.16}$$

According to the transformation laws (0.13) and (3.7) and the transformation law for $\star\Omega$ which, as follows from the homomorphism property of $\star$, is the same as for Ω, we see that equations (3.16) are gauge invariant. The correspondence between (3.12), (3.15) and the system (3.16) follows from direct calculations.

Let us now try to define the deformed Lagrangian of the gauge field according to the following formula:

$$L\tau^5 = \Omega \wedge \star\Omega = \star\Omega \wedge \Omega. \tag{3.17}$$

According to (1.18) and the fact that τ^5 as well as $\star\Omega \wedge \Omega$ are Hermitean differential forms, the operator L is also Hermitean. From (1.18) and the gauge transformation law (3.7) for Ω and $\star\Omega$ a similar formula for L follows,

$$\tilde{L} = ULU^*. \tag{3.18}$$

Now, in order to find a gauge invariant action from L, we have by analogy with the undeformed case to take some integral over $\mathcal{M}_\kappa$. As in the case of defining the Hilbert space corresponding to D_κ we have no recipe for how to do this, and we may only make some general statements. We propose that a linear subspace $L^1(\mathcal{M}_\kappa)$ of $\mathcal{M}_\kappa$ must exist and a positive linear functional $h : L^1(\mathcal{M}_\kappa) \to \mathbb{C}$. It is natural to suppose that $L^1(\mathcal{M}_\kappa)$ is invariant under the $\mathcal{P}_\kappa$-coaction (1.3),

$$\Phi_{\mathrm{R}}(L^1(\mathcal{M}_\kappa)) = L^1(\mathcal{M}_\kappa) \otimes \mathcal{P}_\kappa. \tag{3.19}$$

It is also natural to suppose that the functional h is $\mathcal{P}_\kappa$ invariant so that for every $a \in L^1(\mathcal{M}_\kappa)$

$$(h \otimes \mathrm{id}) \circ \triangle(a) = h(a)1_{\mathcal{P}_\kappa}. \tag{3.20}$$

Now let $U_{\mathcal{M}_\kappa}$ be the group of all $U \in \mathcal{M}_\kappa$ satisfying (0.11) and $U_{\mathcal{M}_\kappa,h}$ the subgroup of $U_{\mathcal{M}_\kappa}$ additionally preserving h, so that for every $a \in L^1(\mathcal{M}_\kappa)$ and $U \in U_{\mathcal{M}_\kappa,h}$

$$UaU^* \in L^1(\mathcal{M}_k) \tag{3.21}$$

and

$$h(UaU^*) = h(a). \tag{3.22}$$

We see now that the invariance gauge group of equations (3.2) and (3.16) is $U_{\mathcal{M}_\kappa}$; however, the action may be invariant only on the action of $U_{\mathcal{M}_\kappa,h}$.

An additional spin-0 field A_4 will be scalar if in the Dirac operator (2.7) $\gamma^4 = \lambda I$ and pseudoscalar if $\gamma^4 = \lambda\gamma^5$. The natural appearance of such a field in a non-commutative situation lies in accordance with the [1, 2] approach and seems very important.

4. Conclusions

In this paper we have defined the Dirac operator on κ-Minkowski space according to the Connes scheme. In the special case it coincides with that proposed in [11]. We also constructed the deformed Maxwell equations and deformed Lagrangian for electrodynamics on κ-Minkovski space, and mentioned the natural appearance of the spin-0 gauge field in the theory. Since almost all of the main constructions used in this paper, including the proof of commutativity of the diagram (0.8) (which follows from general formulae (1.22) and (2.5)), the geometric form of the deformed Maxwell equation (3.16) and the expression (3.17) for the deformed Lagrangian, have a very general form our approach probably can be applied to many other interesting examples of quantum spaces. In a forthcoming paper we shall study in this framework the field theory on the $SU_q(2)$ quantum group considered as a quantum manifold.

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